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Radial Fredholm perturbation in the two-dimensional Ising model and gap-exponent relation

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Abstract. We consider concentric circular defects in the two-dimensional Ising model, which are distributed according to a generalized Fredholm sequence, i.e. at exponentially increasing radii. This type of aperiodicity does not change the bulk critical behaviour but introduces a marginal extended perturbation. The critical exponent of the local magnetization is obtained through finite-size scaling, using a corner transfer matrix approach in the extreme anisotropic limit. It varies continuously with the amplitude of the modulation and is closely related to the magnetic exponent of the radial Hilhorst–van Leeuwen model. Through a conformal mapping of the system onto a strip, the gap-exponent relation is shown to remain valid for such an aperiodic defect.

1. Introduction

The influence of a layered aperiodic modulation of the couplings on the critical behaviour of the two-dimensional Ising model has been much studied recently (see [1] and references therein). Such a perturbation may be relevant, marginal or irrelevant, depending on the sign of the crossover exponent [1, 2]

$$\Phi = 1 + \nu(\omega - 1) \quad (1.1)$$

which involves the bulk correlation length exponent ν and the wandering exponent of the aperiodic sequence ω [3, 4].

In the case of a marginal perturbation, $\Phi = 0$, some exact results have been obtained for the Ising model. Critical exponents are found to vary continuously with the modulation amplitude [1, 2, 5–8] and, when the critical coupling is shifted, the bulk critical point becomes strongly anisotropic, i.e. the exponents of the correlation length, parallel and perpendicular to the layers, are different and their ratio varies continuously with the amplitude of the modulation [8].

For some aperiodic sequences, the defect density vanishes in the thermodynamic limit, there is no shift in the critical coupling and the bulk critical properties remain unchanged. The Fredholm sequence [9], which belongs to this class and leads to a marginal perturbation for the layered Ising model, has recently been studied on a semi-infinite system [7]. Continuously varying surface exponents were obtained and it was shown that this type

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of aperiodic perturbation, which happens to be very regular, can be considered as a discrete realization of the Hilhorst–van Leeuwen (HvL) model [10], i.e. a semi-infinite Ising model with smoothly varying couplings.

In the present paper, we study the critical properties of a radial aperiodic perturbation in the two-dimensional Ising model. Instead of the parallel line defects of the layered system, we consider concentric circular defects distributed according to the Fredholm sequence. This type of perturbation is closely related to the radial HvL defect [11–13].

Our main motivation is to check the validity of the gap-exponent relation [14, 15] which follows from the transformation of the critical correlation functions under the conformal mapping of the original system onto a strip [16]. Since conformal transformations cannot be used on strongly anisotropic systems [17], such a relation is excluded in the case of bulk aperiodic perturbations. But it is known to be satisfied in some marginally inhomogeneous systems at the critical point, provided the inhomogeneity is properly transformed, as shown for a narrow-line defect [18, 19] and later for extended defects of the HvL type [20, 21, 11, 12].

The structure of the paper is the following. In section 2, we present the model and recall the properties of the Fredholm sequence. In section 3, the local magnetization is studied by the corner transfer matrix method of Peschel and Wunderling [13] and the local critical exponent is compared to the value for the HvL defect. In section 4, the validity of the gap-exponent relation is discussed and our conclusions are given in section 5.

2. The radial Fredholm perturbation

We consider, in the (ρ, ϑ) -plane, a two-dimensional Ising model with Hamiltonian

$$-\beta H = -\beta H_c + g \sum_{p=-\infty}^{+\infty} \int \varepsilon(\rho, \vartheta) \delta(\rho - \rho_p) \rho \, d\rho \, d\vartheta \quad \rho_p = m^p \quad (2.1)$$

in the continuum limit. Here H_c is the critical Hamiltonian of the unperturbed system, ε is the energy density operator and the energy perturbation, with an amplitude g per unit length, is located on circles with radii $\rho_p = m^p$.

The contribution of negative values of p to the perturbation is irrelevant since it renormalizes to a point defect with a finite amplitude. The local critical behaviour, which is governed by the long-distance behaviour of the perturbation, remains unaffected if one considers circles with radii distributed according to the generalized Fredholm sequence [7], i.e. with $\rho_p = m^p + 1$, $p \geq 0$ and integer values of $m > 1$.

This aperiodic sequence, which is the characteristic sequence of the powers of m , follows from the substitution on the three letters A , B and C :

$$\begin{aligned} A &\rightarrow S(A) = A B C C \dots C \\ B &\rightarrow S(B) = B C C C \dots C \\ C &\rightarrow S(C) = \underbrace{C C C C \dots C}_m. \end{aligned} \quad (2.2)$$

With words of length $m = 2$, one recovers the usual Fredholm substitution [9]. Starting the substitution with A and associating to the k th letter in the sequence the digit $f_k = 0$ for A

and C , and $f_k = 1$ for B , after n iterations and with $m = 2$, one obtains:

$n = 0$	A															
$n = 1$	A	B														
$n = 2$	A	B	B	C												
$n = 3$	A	B	B	C	B	C	C	C								
$n = 4$	A	B	B	C	B	C	C	C	B	C	C	C	C	C	C	C
f_k	0	1	1	0	1	0	0	0	1	0	0	0	0	0	0	0

which gives $f_k = 1$ when $k = 2^p + 1$.

The substitution matrix has eigenvalues $m, 1, 1$ [7], so that the wandering exponent ω vanishes, leading to a marginal layered perturbation according to (1.1). It is easy to verify that the perturbation is also marginal with a radial defect. The total perturbation inside a circle with radius $R \gg 1$ is obtained by summing over the contributions of the circles up to $p_{max} \approx \ln R / \ln m$

$$2\pi g \sum_{p=0}^{p_{max}} m^p \approx \frac{2\pi mgR}{m-1} \tag{2.4}$$

and the average perturbation per unit surface at a length scale R is given by

$$\bar{g}(R) \sim \frac{g}{R} \tag{2.5}$$

Under rescaling by a factor $b = R/R'$, this thermal perturbation transforms according to

$$\frac{g'}{R'} = b^{1/\nu} \frac{g}{R} \tag{2.6}$$

so that, with $\nu = 1$ in the two-dimensional Ising model, g is scale invariant, i.e. the radial aperiodic perturbation is marginal. Furthermore, according to (2.5), the average perturbation density vanishes in the thermodynamic limit, leaving the bulk critical point unchanged.

3. Corner transfer matrix study of the local magnetization

In this section, the critical behaviour of the local magnetization at the centre of the defect will be deduced from a finite-size scaling analysis at the bulk critical point. This can be achieved using the corner transfer matrix method of Peschel and Wunderling [13] which allows a study of rotation-symmetric defects, while working on a lattice.

We first consider the sector of a lattice Ising model shown in figure 1. With the same lattice parameter $a_1 = a_2 = a$ in both directions, the opening angle is $\theta = \pi/2$. There are

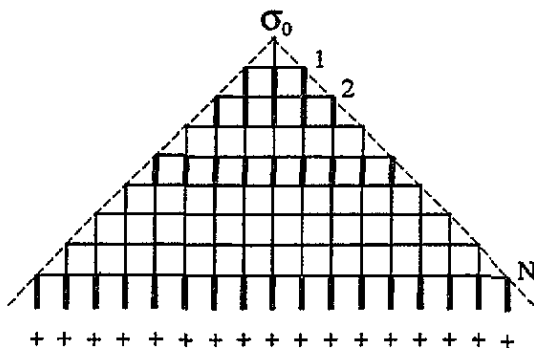


Figure 1. Sector of the anisotropic Ising square lattice which is used to construct the corner transfer matrix. The horizontal couplings are constant and equal to K_1 . The vertical couplings follow the Fredholm sequence and are equal either to K_2 or to rK_2 (full curves). The Ising spins on the last row are held fixed in order to calculate the local magnetization.

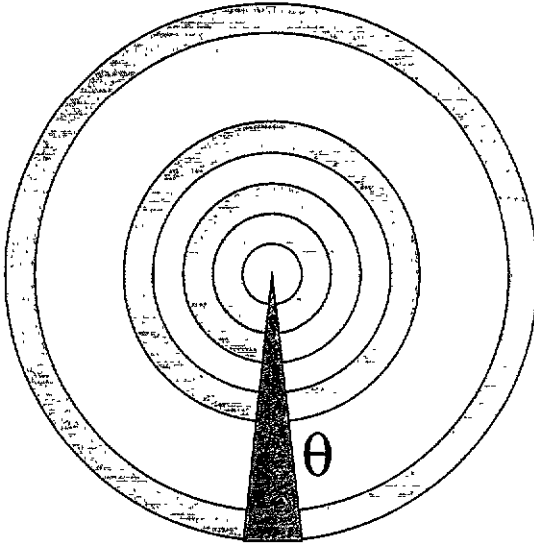


Figure 2. Through a rescaling of the lattice parameters, one restores isotropy near the critical point. The sector of figure 1 (full triangle) now has a vanishing opening angle θ . Covering the plane with such sectors, one generate the radial Fredholm defect (shaded circles).

$N + 1$ horizontal rows with fixed boundary conditions on the last one. The interactions are between first-neighbour spins, constant and equal to K_1 in the horizontal direction, while they are given by $K_2(k)$ and depend on the row index, $k = 0, N$, in the vertical direction. This dependence follows the generalized Fredholm sequence, so that the vertical couplings take the form

$$K_2(k) = K_2 r^{fk+1} \tag{3.1}$$

where r is the modulation factor and the f_k s are the digits defined in (2.3).

Let us take the extreme anisotropic limit, $K_2 \rightarrow 0$ and $K_1 \rightarrow \infty$, while keeping the ratio $\lambda = K_2/K_1^*$ fixed ($K_1^* = -\frac{1}{2} \ln \tanh K_1 \rightarrow 0$ is a dual coupling). On the square lattice, the correlation lengths become different in the two directions with $\xi_2/\xi_1 \simeq 2K_1^* \ll 1$ near the critical point. In order to recover an isotropic system which properly describes the continuum problem of the last section, one has to rescale the lattice parameters such that $\xi_1 a_1 = \xi_2 a_2$, which gives $a_1 = 2K_1^* \rightarrow 0$ if one takes a_2 for the unit length. The opening angle of the sector is now reduced to $\theta = 4K_1^*$ and the number of sectors needed to cover the plane

$$n = \frac{\pi}{2K_1^*} \tag{3.2}$$

goes to infinity in the extreme anisotropic limit.

In this way, as shown in figure 2, the rotation symmetry of the original problem is restored and the system becomes continuous along the defects. The perturbation per unit length is $(r - 1)K_2/a_1$ which allows us to identify the continuum parameter

$$g = \frac{1}{2}(r - 1) \tag{3.3}$$

at the critical point $\lambda_c = 1$.

In the extreme anisotropic limit, the Baxter corner transfer matrix \mathcal{T} [22] takes the simple form

$$\mathcal{T} = \exp\left(-\frac{1}{2}\theta\mathcal{H}\right) \tag{3.4}$$

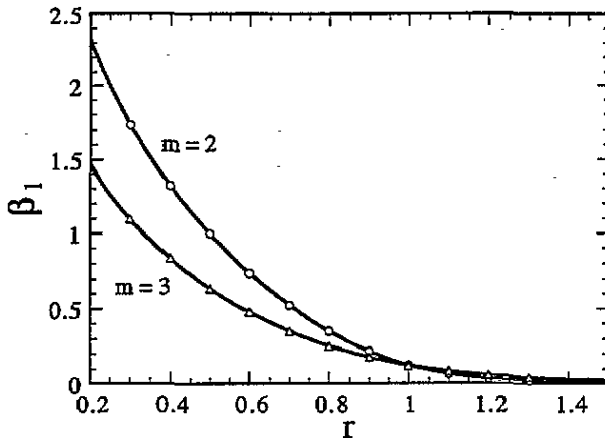


Figure 3. Variation of the local magnetization exponent β_l with the modulation factor r for the generalized Fredholm sequence with $m = 2$ and $m = 3$. The curves correspond to the conjectured expression of (3.9) and the points are finite-size scaling estimates.

where θ is the infinitesimal opening angle of the sector and \mathcal{H} is the Hamiltonian of the inhomogeneous quantum Ising chain:

$$\mathcal{H} = -\frac{1}{2} \sum_{k=1}^{N-1} 2k\sigma_k^z - \frac{1}{2} \sum_{k=0}^{N-1} (2k+1)\lambda(k)\sigma_k^x\sigma_{k+1}^x \quad \lambda(k) = \lambda r^{k+1}. \quad (3.5)$$

The σ s are Pauli spin operators and the coupling $\lambda(k)$ has the same aperiodic modulation as K_2 in 3.1.

The magnetization of the central spin, given by

$$m_0 = \frac{\text{Tr}(\sigma_0^x \sigma_N^x T^n)}{\text{Tr}(T^n)} \quad (3.6)$$

can be re-expressed in terms of the Fermi operators which diagonalize the corner transfer matrix [23, 24]. In this way, one obtains the local magnetization as a product

$$m_0 = \prod_{\nu=1}^N \tanh\left(\frac{1}{2}\pi\epsilon_\nu\right) \quad (3.7)$$

where the ϵ_ν s are the N non-vanishing diagonal fermion excitations of the quantum chain† (see [13] or appendix B in [25] for details).

The excitations of the Hamiltonian (3.5) at the critical point $\lambda_c = 1$ have been obtained numerically on chains with $N = m^p + 1$ spins with $p = 4$ to 16 for $m = 2$ and $p = 3$ to 9 for $m = 3$. The modulation factor r has been varied from 0.3 to 1.3 with steps of 0.1. The critical value of m_0 on a finite system vanishes as N^{-x_l} , where the scaling dimension of the local magnetization, x_l , is continuously varying in the present case. This exponent is also equal to the local magnetization exponent β_l since $\nu = 1$ in the 2D Ising model. The finite-size estimates for β_l obtained from sequence extrapolations using the BST algorithm [26] are shown in figure 3.

The layered Fredholm modulation on a semi-infinite system is known to lead to the same critical behaviour as the HvL model with couplings $K_2(k) = K_2(1 + \alpha/k)$ if one makes the correspondence [7]

$$\alpha \rightarrow \frac{\ln r}{\ln m}. \quad (3.8)$$

† The lowest excitation ϵ_0 vanishes due to the fixed boundary conditions.

Table 1. Extrapolated finite-size estimates for the local magnetization exponent β_l and the shift Δ contributed by the localized mode when $r < 1$, as functions of the modulation factor r for the Fredholm radial defect with $m = 2$ and $m = 3$. The figures in brackets give the estimated uncertainty in the last digit. Each second column gives the expected values of (3.9).

r	$m = 2$				$m = 3$			
	β_l		Δ		β_l		Δ	
0.3	1.7373	1.7370	1.736 97(6)	1.736 965 59	1.0973	1.0963	1.095 903 1(5)	1.095 903 27
0.4	1.3221	1.3220	1.321 928 11(4)	1.321 928 09	0.8373	0.8356	0.834 043(1)	0.834 043 77
0.5	1.0010	1.0006	0.999 999 9(1)	1	0.6382	0.6362	0.630 929(2)	0.630 929 75
0.6	0.7405	0.7398	0.736 964(3)	0.736 965 59	0.4801	0.4780	0.464 93(6)	0.464 973 52
0.7	0.5254	0.5245	0.514 62(7)	0.514 573 17	0.3538	0.3522	0.326(2)	0.324 659 52
0.8	0.3505	0.3498	0.321 91(5)	0.321 928 09	0.2546	0.2537	0.211(6)	0.203 114 01
0.9	0.2168	0.2166	0.1517(3)	0.152 003 09	0.1795	0.1793	0.101(4)	0.095 903 27
1.0	0.1250	0.125	—	—	0.1250	0.125	—	—
1.1	0.0692	0.0690	—	—	0.0870	0.0868	—	—
1.2	0.0382	0.0377	—	—	0.0611	0.0605	—	—
1.3	0.0215	0.0208	—	—	0.0436	0.0425	—	—

Assuming the same relation for the radial defect and using the analytical result of reference [13] for the HvL model, we are led to the following conjecture for the Fredholm exponent†

$$\beta_l = 2 \left(\frac{\ln r}{\ln m} \right)^2 \int_0^\infty du \frac{\sinh^2 u}{\sinh \left(2\pi \left| \frac{\ln r}{\ln m} \right| \cosh u \right)} \left(+ \left| \frac{\ln r}{\ln m} \right| \right). \quad (3.9)$$

Here the quantity in brackets is a shift Δ , contributed by a localized mode, which has to be added when $r < 1$. Our numerical estimates for β_l are in reasonable agreement with this expression as shown in table 1. The uncertainties on β_l , estimated by comparing different extrapolated values given by the BST algorithm, are much smaller than the actual deviations from the conjectured values. Such a behaviour is known to occur when logarithmic corrections to scaling are present [26]. The correspondence (3.8) is strongly supported by the extrapolated values of the shift Δ for which the agreement with the expected values is excellent.

4. Conformal mapping and gap-exponent relation

Let us now consider the transformation of the marginal perturbation in the continuum limit (equation (2.1)), under the logarithmic conformal mapping [16]

$$w = \frac{L}{2\pi} \ln z \quad w = u + iv \quad z = \rho e^{i\vartheta} \quad (4.1)$$

which transforms the whole plane into a strip ($-\infty < u < +\infty$, $0 < v < L$) with periodic boundary conditions in the v -direction. The local dilatation factor is $b(z) = |dw/dz|^{-1} = 2\pi\rho/L$ and the amplitude of the thermal perturbation, $t(z) = g \sum_p \delta(\rho - \rho_p)$, is changed into

$$\begin{aligned} t(w) &= b(z)^{1/\nu} t(z) = g \left(\frac{2\pi\rho}{L} \right)^{1/\nu} \sum_{p=-\infty}^{+\infty} \delta \left(\exp \frac{2\pi u}{L} - m^p \right) \\ &= g \left(\frac{2\pi}{L} \right)^{1/\nu} \sum_{p=-\infty}^{+\infty} \delta \left\{ \exp \left[\frac{2\pi u}{L} \left(1 - \frac{1}{\nu} \right) \right] - \exp \left[p \ln m - \frac{2\pi u}{\nu L} \right] \right\}. \end{aligned} \quad (4.2)$$

† The amplitude α is half the one used in [13].

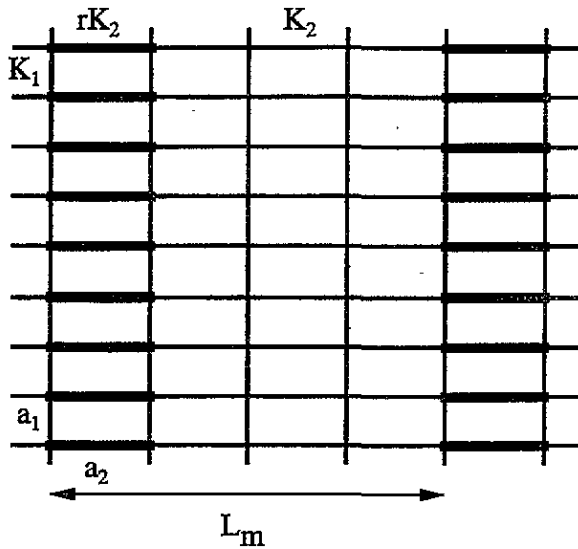


Figure 4. The logarithmic conformal mapping transforms the radial aperiodic Fredholm defect in the plane into a periodic defect on the strip. The period L_m is proportional to the strip width L .

With the Ising model, $\nu = 1$, the perturbation is marginal, and the first part of the u -dependence is eliminated so that

$$\begin{aligned}
 t(w) &= g \left(\frac{2\pi}{L} \right) \sum_{p=-\infty}^{+\infty} \delta \left\{ 1 - \exp \left[-\frac{2\pi}{L} \left(u - p \frac{L \ln m}{2\pi} \right) \right] \right\} \\
 &= g \sum_{p=-\infty}^{+\infty} \delta \left(u - p \frac{L \ln m}{2\pi} \right). \tag{4.3}
 \end{aligned}$$

As shown in figure 4, the perturbation now consists of straight line defects, the distance between two successive lines

$$L_m = L \ln m / 2\pi \tag{4.4}$$

being proportional to the width of the strip L . The discrete dilatation invariance of the radial defect on the plane has been transformed into a discrete translation invariance along the strip.

In the case of the HvL radial defect, the continuous dilatation invariance of the perturbation, $t(z) = g/\rho$ on the plane, leads to a constant deviation from the critical coupling $t(w) = g(2\pi/L)$ on the strip [11, 12, 25]. If ξ_ϕ denotes the correlation length associated with the spin-spin or the energy-energy correlations on the off-critical isotropic strip, it satisfies the following finite-size scaling relation [25]:

$$\xi_\phi^{-1}(t, L) = L^{-1} X_\phi(cL^{1/\nu}t) \tag{4.5}$$

where the gap scaling function $X_\phi(\tau)$ is universal [27], t is the deviation from the bulk critical temperature and c is a non-universal constant. The defect scaling dimension follows from the gap-exponent relation [14–16] with

$$x_l^\phi = \frac{L}{2\pi} \xi_\phi^{-1}(t, L) = \frac{1}{2\pi} X_\phi(cL^{1/\nu}t). \tag{4.6}$$

It is continuously varying with the defect amplitude g , as expected for a marginal perturbation, but it varies in a universal way: different models belonging to the same class of universality will show the same dependence.

With the Fredholm radial defect, which gives a periodic layered perturbation in the strip geometry, one could directly deduce the gap (inverse correlation length) from the appropriate transfer matrix on the strip and use (4.6). Alternatively, one may estimate the deviation t from the bulk critical temperature in (4.6) which is associated with the periodic defect. A measure of this deviation is provided by the shift in the critical temperature induced by the same perturbation (i.e. parallel line defects with a fixed distance L_m) on the infinite plane.

In order to avoid the calculation of the non-universal constant c , we shall proceed via a comparison to the HvL problem. We use the same extreme anisotropic limit as for the corner transfer matrix, with vertical couplings $K_1 \rightarrow \infty$, horizontal couplings $K_2 \rightarrow 0$ and vertical ladder defects corresponding to modified couplings

$$K'_2 = rK_2 \quad (4.7)$$

as shown in figure 4. As above, a length rescaling ($a_2 = 1$, $a_1 = 2K_1^*$) is understood in order to restore isotropy. For the radial HvL problem the perturbed interactions $K_2(k) = K_2(1 + \alpha/k)$ lead to a homogeneous shift of the horizontal couplings with

$$K'_2 = K_2 \left(1 + \alpha \frac{2\pi}{L} \right). \quad (4.8)$$

The critical temperature of a periodic layered anisotropic system follows from the relation [28]

$$\sum_j K_2^*(j) = \sum_j K_1(j) \quad (4.9)$$

where K_2 -bonds are perpendicular to the layers and the sums are over a period. In the case of the Fredholm problem, this leads to

$$(L_m - 1)K_2^* + K_2'^* = L_m K_1 \quad (4.10)$$

where, in the extreme anisotropic limit, equation (4.7) leads to

$$K_2'^* = K_2^* - \frac{1}{2} \ln r. \quad (4.11)$$

Putting (4.11) into (4.10) gives the criticality condition

$$K_2^* - \frac{\ln r}{2L_m} = K_1. \quad (4.12)$$

The corresponding relation for the HvL problem is obtained using (4.10) with $L_m = 1$. Since the modified coupling in (4.8) corresponds to $r = 1 + \alpha(2\pi/L)$, with $L \gg 1$ equation (4.11) gives

$$K_2'^* \approx K_2^* - \alpha \frac{\pi}{L} \quad (4.13)$$

and finally the criticality condition reads

$$K_2^* - \alpha \frac{\pi}{L} = K_1. \quad (4.14)$$

Comparing equations (4.12) and (4.14) and using (4.4) leads to the conjectured correspondence of (3.8).

Besides the local magnetization exponent of (3.9), the gap-exponent relation also gives the local energy exponent of the radial Fredholm defect as [12]

$$x_i^e = 1 + 2 \left(\frac{\ln r}{\ln m} \right)^2 + O \left[\left(\frac{\ln r}{\ln m} \right)^4 \right] \quad (4.15)$$

where, due to the self-duality of the Ising model, only even powers of $\ln r / \ln m$ enter the expansion [29].

5. Conclusion

The close relationship between the discrete aperiodic Fredholm defect and the continuous HvL inhomogeneity in the two-dimensional Ising model has been established in a study of the surface critical behaviour of the layered system in [7]. It has also been recently verified for the bulk layered Fredholm defect [30].

In the present paper the validity of this connection, which is summarized in (3.8), is further extended to radially symmetric defects. The inhomogeneity in this case corresponds to an infinite sequence of concentric circles with exponentially increasing radii. The separation of distant circles becoming very large, the density of defect bonds goes to zero, thus the bulk critical behaviour of the system remains unchanged. However, an infinite sequence of defect circles modifies the local critical behaviour at the centre of the defect.

Our second observation, concerning the validity of the gap-exponent relation for the Fredholm defect, is somewhat unexpected. It was known until now [2] that some aspects of conformal invariance, including the gap-exponent relation, are still satisfied for some marginally inhomogeneous systems, which either contain a finite number of defect lines or display a smooth variation. In the Fredholm problem the number of defect circles is infinite, furthermore, the perturbation changes the continuous dilatation symmetry into a discrete one.

A similar analysis of the layered Fredholm defect problem [7, 30] reveals that the gap-exponent relation stays valid in this case, too. Now the transformed perturbation on the strip is still periodic with period L_m as given in (4.4), however its shape is much more complicated than for the radial defect. Taking the anisotropic limit and considering the product of L_m successive transfer matrices one can establish the same relation with the HvL model as in (3.8). These observations lead us to conjecture the validity of the gap-exponent relation for any marginal perturbation which does not modify the bulk critical behaviour of the system.

If the marginal perturbation extends over the volume of the system with a non-vanishing density, the gap-exponent relation is generally no longer valid. As mentioned in the introduction, this type of marginal aperiodic perturbations lead to strongly anisotropic systems [8] which cannot be transformed using conformal techniques.

Acknowledgments

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